

PART III - MATHEMATICS

SECTION - I (Total Marks : 21)
(Single Correct Answer Type)

This section contains 7 multiple choice questions. Each question has four choices (A), (B), (C) and (D) out of which ONLY ONE is correct.

47. Let $P = \{\theta : \sin \theta - \cos \theta = \sqrt{2} \cos \theta\}$ and $Q = \{\theta : \sin \theta + \cos \theta = \sqrt{2} \sin \theta\}$ be two sets. Then
 (A) $P \subset Q$ and $Q - P \neq \emptyset$ (B) $Q \not\subset P$ (C) $P \not\subset Q$ (D) $P = Q$

47. (D) $P : \sin \theta - \cos \theta = \sqrt{2} \cos \theta \Rightarrow \tan \theta = 1 + \sqrt{2}$

$$Q : \sin \theta + \cos \theta = \sqrt{2} \sin \theta \Rightarrow \tan \theta = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$$

$$\therefore P = Q.$$

48. Let the straight line $x = b$ divide the area enclosed by $y = (1-x)^2$, $y = 0$, and $x = 0$ into two parts

R_1 ($0 \leq x \leq b$) and R_2 ($0 \leq x \leq 1$) such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

- (A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

48. (B) $R_1 - R_2 = \frac{1}{4} \Rightarrow \int_0^b (1-x)^2 dx - \int_b^1 (1-x)^2 dx = \frac{1}{4} \Rightarrow \frac{1}{3} - \frac{2(1-b)^3}{3} = \frac{1}{4}$

$$\Rightarrow (1-b)^3 = \frac{1}{8} \Rightarrow b = \frac{1}{2}.$$

49. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of

$\frac{a_{10} - 2a_8}{2a_9}$ is

- (A) 1 (B) 2 (C) 3 (D) 4

49. (C) Since $\alpha^2 - 6\alpha - 2 = 0 \Rightarrow \alpha^2 - 2 = 6\alpha$
 and $\beta^2 - 6\beta - 2 = 0 \Rightarrow \beta^2 - 2 = 6\beta$

$$\frac{a_{10} - 2a_8}{2a_9} = \frac{(\alpha^{10} - \beta^{10}) - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)} = \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{2(\alpha^9 - \beta^9)} = \frac{\alpha^8 \cdot 6\alpha - \beta^8 \cdot 6\beta}{2(\alpha^9 - \beta^9)} = 3.$$

50. A straight line L through the point $(3, -2)$ is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the x -axis, then the equation of L is

(A) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$ (B) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
 (C) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$ (D) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

50. (B) If required line has slope = m

$$\Rightarrow \quad \left| \frac{m + \sqrt{3}}{1 - \sqrt{3}m} \right| = \tan 60^\circ \quad \Rightarrow \quad m = 0, \sqrt{3}$$

Since, the line intersects the x-axis

$$\Rightarrow \text{ equation is, } (y+2) = \sqrt{3} (x-3) \Rightarrow y - \sqrt{3}x + 2 + 3\sqrt{3} = 0.$$

51. Let (x_0, y_0) be the solution of the following equations

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$

$$3^{\ln x} = 2^{\ln y}$$

Then x_0 is

- (A) $1/6$ (B) $1/3$ (C) $1/2$ (D) 6

51. (C) Since $(\ln 2)(\ln 2 + \ln x) = (\ln 3)(\ln 3 + \ln y)$ and $\ln x \cdot \ln 3 - \ln y \cdot \ln 2 = 0$

$$\Rightarrow (\ln 2) \cdot (\ln x) - \ln y \cdot \ln 3 = (\ln 3)^2 - (\ln 2)^2 \quad \dots \dots \dots \text{(i)}$$

$$\text{and } \ln x \cdot \ln 3 - \ln y \cdot \ln 2 = 0 \quad \dots \dots \dots \text{(ii)}$$

Using (i) $\times \ln 2 -$ (ii) $\times \ln 3$, we get

$$\ln x \cdot \{(\ln 2)^2 - (\ln 3)^2\} = (\ln 2) \{(\ln 3)^2 - (\ln 2)^2\}$$

$$\Rightarrow \ln x = -\ln 2 \Rightarrow x = 1/2.$$

52. The value of $\int_{\frac{\sqrt{\ln 3}}{\sqrt{\ln 2}}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$ is

- $$52. \quad (\text{A}) \quad I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$$

$$x^2 = t \quad \Rightarrow \quad 2x \, dx = dt$$

$$\Rightarrow I = \frac{1}{2} \int_{\ln 2}^{\ln 3} \frac{\sin t}{\sin t + \sin(\ln 6 - t)} dt$$

$$\text{Using } \int_a^b f(x)dx = \frac{1}{2} \int_a^b (f(x) + f(a+b-x))dx$$

$$\Rightarrow I = \frac{1}{4} \int_{\ln 2}^{\ln 3} \left(\frac{\sin t}{\sin t + \sin(\ln 6 - t)} + \frac{\sin(\ln 6 - t)}{\sin(\ln 6 - t) + \sin t} \right) dt \quad [\text{since } \ln 3 + \ln 2 = \ln 6]$$

$$\Rightarrow I = \frac{1}{4} \int_{\ln 2}^{\ln 3} dt = \frac{1}{4} [t]_{\ln 2}^{\ln 3} = \frac{1}{4} \ln \frac{3}{2}.$$

53. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$, and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector \vec{v} in the plane of \vec{a} and \vec{b} , whose projection on \vec{c} is $1/\sqrt{3}$ is given by
 (A) $\hat{i} - 3\hat{j} + 3\hat{k}$ (B) $-3\hat{i} - 3\hat{j} - \hat{k}$ (C) $3\hat{i} - \hat{j} + 3\hat{k}$ (D) $\hat{i} + 3\hat{j} - 3\hat{k}$

53. (C) $\vec{v} = \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - \hat{j} + \hat{k})$

$$\frac{\vec{v} \cdot \vec{c}}{|\vec{c}|} = \frac{1}{\sqrt{3}} \Rightarrow (\lambda + \mu) - (\lambda - \mu) - (\lambda + \mu) = 1 \Rightarrow \mu = \lambda + 1$$

$$\vec{v} = \hat{i}(2\lambda + 1) - \hat{j} + \hat{k}(2\lambda + 1)$$

Only option (C) satisfies this one.

SECTION - II (Total Marks : 16)

(Multiple Correct Choice Type)

This section contains 4 multiple choice questions. Each question has four choices (A), (B), (C) and (D) out of which ONE OR MORE may be correct.

54. Let $f: R \rightarrow R$ be a function such that $f(x+y) = f(x) + f(y)$, $\forall x, y \in R$.

If $f(x)$ is differentiable at $x = 0$, then

- (A) $f(x)$ is differentiable only in a finite interval containing zero
- (B) $f(x)$ is continuous $\forall x \in R$
- (C) $f'(x)$ is constant $\forall x \in R$
- (D) $f(x)$ is differentiable except at finitely many points

54. (B)(C)
$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) + f(h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

$$\text{Put } x = y = 0, \quad f(0) = 0$$

$$f'(x) = f'(0)$$

$$f(x) = f'(0)x + c$$

$$f(x) = kx + c, c = 0 \text{ as } f(0) = 0$$

\therefore Function is continuous. $f'(x)$ is constant.

55. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$.

If the hyperbola passes through a focus of the ellipse, then

- (A) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
- (B) a focus of the hyperbola is $(2, 0)$
- (C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
- (D) the equation of the hyperbola is $x^2 - 3y^2 = 3$

55. (B)(D)
$$\frac{x^2}{4} + \frac{y^2}{1} = 1, \quad a^2 = 4, b^2 = 1, e^2 = 1 - (b^2/a^2) \Rightarrow e = \sqrt{3}/2$$

Eccentricity of hyperbola $= 2/\sqrt{3}$.

Focus of ellipse $\equiv (\pm ae, 0) = (\pm \sqrt{3}, 0)$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ passes through focus } (\pm \sqrt{3}, 0)$$

$$\therefore a^2 = 3 \quad \therefore b^2 = a^2(e^2 - 1) = 1$$

$$\therefore \text{Equation of hyperbola: } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{3} - \frac{y^2}{1} = 1$$

$$x^2 - 3y^2 = 3$$

Focus of hyperbola $\equiv (\pm ae, 0) = (\pm 2, 0)$.

56. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P , then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to
 (A) M^2 (B) $-N^2$ (C) $-M^2$ (D) MN

56. (C) Since $M^T = -M$, $N^T = -N$

$$\begin{aligned} & M^2N^2(M^TN)^{-1}(MN^{-1})^T \\ &= M^2N^2(-MN)^{-1}(MN^{-1})^T = M^2N^2(N^{-1} \cdot (-M)^{-1}) \cdot (N^T)^{-1} \cdot (-M) \\ &= M^2N(-M)^{-1} \cdot (-N)^{-1} \cdot (-M) = M(MN)(MN)^{-1}(-M) = -M^2. \end{aligned}$$

Note that the statement which is given in the problem is incorrect, as skew matrix of odd order can't be invertible.

57. The vector(s) which is/are coplanar with vectors $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$ is / are
 (A) $\hat{j} - \hat{k}$ (B) $-\hat{i} + \hat{j}$ (C) $\hat{i} - \hat{j}$ (D) $-\hat{j} + \hat{k}$

57. (A)(D) Let $\vec{a} = \lambda(\hat{i} + \hat{j} + 2\hat{k}) + \mu(\hat{i} + 2\hat{j} + \hat{k})$
 $= (\lambda + \mu)\hat{i} + (\lambda + 2\mu)\hat{j} + (2\lambda + \mu)\hat{k}$
 $\vec{a} \cdot (\hat{i} + \hat{j} + \hat{k}) = 0 \quad \therefore \quad \lambda = -\mu \quad \therefore \quad \vec{a} = \lambda(-\hat{j} + \hat{k})$

Taking $\lambda = 1$, $\vec{a} = -\hat{j} + \hat{k}$

$\lambda = -1$, $\vec{a} = \hat{j} - \hat{k}$.

SECTION - III (Total Marks : 15)

(Paragaph Type)

This Section contains **2 paragraphs**. Based upon one of the paragraphs **2 multiple choice questions** and based on other paragraph **3 multiple choice questions** have to be answered. Each of these has four choices (A), (B), (C) and (D) out of which **ONLY ONE** is correct.

Paragraph for question Nos 58 and 59

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2 . However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2 . Now 1 ball is drawn at random from U_2 .

58. The probability of the drawn ball from U_2 being white is
 (A) $13 / 30$ (B) $23 / 30$ (C) $19 / 30$ (D) $11 / 30$

$$58. \text{ (B)} \quad P = \frac{1}{2} \left[\underbrace{\frac{3}{5} \times 1 + \frac{2}{5} \times \frac{1}{2}}_{\text{For appearing head}} \right] + \frac{1}{2} \left[\underbrace{\frac{^3C_1 \times ^2C_1}{5C_2} \times \frac{2}{3} + \frac{^3C_2}{5C_2} \times 1 + \frac{^2C_2}{5C_2} \times \frac{1}{3}}_{\text{For appearing tail}} \right] = 23 / 30.$$

59. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is
 (A) $17 / 23$ (B) $11 / 23$ (C) $15 / 23$ (D) $12 / 23$
59. (D) Using Bayes' theorem,

$$P = \frac{\frac{1}{2} \left[\frac{3}{5} \times 1 + \frac{2}{5} \times \frac{1}{2} \right]}{\frac{1}{2} \left[\frac{3}{5} \times 1 + \frac{2}{5} \times \frac{1}{2} \right] + \frac{1}{2} \left[\frac{^3C_1 \times ^2C_1}{5C_2} \times \frac{2}{3} + \frac{^3C_2}{5C_2} \times 1 + \frac{^2C_2}{5C_2} \times \frac{1}{3} \right]} = 12 / 23.$$

Paragraph for question Nos 60 and 62

Let a , b and c be three real numbers satisfying

$$[a \ b \ c] \begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [0 \ 0 \ 0] \quad \dots \dots \dots \text{(E)}$$

60. (D) $D = \begin{vmatrix} 1 & 8 & 7 \\ 9 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0$

\therefore system has non trivial solution.

If $c = k$, $b = -6/7k$, $a = -k/7$

$$\therefore 2a + b + c = 1$$

$$-(2k/7) - (6k/7) + k = 1 \Rightarrow k = -7$$

$$\therefore P(a, b, c) \equiv (1, 6, -7)$$

$$7a + b + c = 7 + 6 - 7 = 6.$$

61. Let ω be a solution of $x^2 - 1 = 0$ with $\operatorname{Im}(\omega) > 0$. If $a = z$ with b and c satisfying (E), then the value

of $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c}$ is equal to

- (A) -2 (B) 2 (C) 3 (D) -3

61. (A) $a = 2, k = -14$

$$c = -14, b = 12, a = 2$$

$$\begin{aligned}
 & \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}} = \frac{3}{\omega^2} + \frac{1}{(\omega^3)^4} + \frac{3(\omega^3)^5}{\omega} \\
 &= \frac{3}{\omega^2} + 1 + \frac{3}{\omega} = \frac{3 + \omega^2 + 3\omega}{\omega^2} = \frac{3(1+\omega) + \omega^2}{\omega^2} = \frac{-3\omega^2 + \omega^2}{\omega^2} = -2
 \end{aligned}$$

62. Let $b = 6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation

$ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} + \frac{1}{\beta} \right)^n$ is

62. (B) $b = 6, k = -7, c = -7, a = 1$

Q.E. becomes $x^2 + 6x - 7 = 0$; $a + b = -6$, $ab = -7$

$$\sum_{n=0}^{\infty} \left(\frac{\alpha + \beta}{\alpha \beta} \right)^n = \sum_{n=0}^{\infty} \left(\frac{6}{7} \right)^n = 1 + \left(\frac{6}{7} \right) + \left(\frac{6}{7} \right)^2 + \dots = \frac{1}{1 - \frac{6}{7}} = 7$$

SECTION - IV (Total Marks : 28) (Integer Answer Type)

This section contains **7 questions**. The answer to each of the questions is a **single-digit integer**, ranging from 0 to 9. The bubble corresponding to the correct answer is to be darkened in the ORS.

63. Let $f: [1, \infty) \rightarrow [(2, \infty)$ be a differentiable function such that $f(1) = 2$, if

$$6 \int_1^x f(t) dt = 3x f(x) - x^3$$

for all $x \geq 1$, then the value of $f(2)$ is

63. 6. Differentiating the given relation,

$$6f(x) = 3f(x) + 3x f'(x) - 3x^2 \Rightarrow f'(x) - \frac{f(x)}{x} = x, \text{ which is in linear form with I.F. } = \frac{1}{x}$$

$$\text{Hence, } f(x) = x^2 + cx \quad \text{with } c = 1. \quad \therefore f(x) = x^2 + x. \quad \therefore f(2) = 6.$$

Note that $f(1)$ is 1 / 3 from the given relation, which is a contradiction.

64. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is

64. 5. Expression $2 \left| z - 3 + \frac{5}{2}i \right|$ which is equivalent to double the distance between complex number z and $(3, -5/2)$ which lies on one the diameter $x = 3$ of circle given. Hence

$$\text{minimum value of expression} = 2 \times \frac{5}{2} = 5.$$

65. Let a_1, a_2, \dots, a_{100} be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^p a_i, 1 \leq p \leq 100$.

For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n , then a_2 is

$$65. 9. \frac{S_m}{S_n} = \frac{a_1 + a_2 + \dots + a_{5n}}{a_1 + a_2 + \dots + a_n} = \frac{\frac{5n}{2}\{2.3 + (5n-1)d\}}{\frac{n}{2}\{2.3 + (n-1)d\}} ;$$

$$\lambda = \frac{5(6 + 5nd - d)}{(6 + nd - d)} \quad [\lambda = \frac{S_m}{S_n}]$$

$\therefore \lambda$ independent of n when $d = 6$

$$\therefore a_2 = a_1 + d = 9.$$

66. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is

66. 2. Extremities of L.R. are $(2, 4)$ & $(2, -4) \Rightarrow \Delta_1 = 6$
 Points of intersection of tangents at these points are $(-2, 0), (1, 3)$ & $(-1, -1) \Rightarrow \Delta_2 = 3$.

67. Let $f(\theta) = \sin \left(\tan^{-1} \left(\frac{\sin \theta}{\sqrt{\cos 2 \theta}} \right) \right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan \theta)} (f(\theta))$ is.

$$\begin{aligned} 67. 1. \quad & \text{Put } k = \tan^{-1} \left(\frac{\sin \theta}{\sqrt{\cos 2 \theta}} \right) \Rightarrow \tan k = \frac{\sin \theta}{\sqrt{\cos 2 \theta}} \\ & \Rightarrow \tan^2 k = \frac{\sin^2 \theta}{1 - 2\sin^2 \theta} \Rightarrow \sec^2 k - 1 = \frac{1}{\cosec^2 \theta - 2} \\ & \Rightarrow \sec^2 k = \frac{\cosec^2 \theta - 1}{\cosec^2 \theta - 2} \Rightarrow \sin^2 k = \frac{1}{\cot^2 \theta} = \tan^2 \theta \\ & \Rightarrow \sin k = \tan \theta \quad \therefore f(\theta) = \sin k = \tan \theta \quad \text{Hence. } \frac{d(f(\theta))}{d(\tan \theta)} = 1 \end{aligned}$$

68. The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$ is

68. 8. Using A.M. \geq G.M.,

$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}}{8} \geq 1$$

\therefore Minimum value of required expression = 8 (for $a = 1$).

69. The positive integer value of $n > 3$ satisfying the equation $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is

$$69. 7. \quad \frac{1}{\sin x} = \frac{1}{\sin 2x} + \frac{1}{\sin 3x} \quad \{\text{where } x = (\pi/n)\}$$

$$\text{or } \frac{\sin 3x - \sin x}{\sin x \cdot \sin 3x} = \frac{1}{\sin 2x} \quad \text{or} \quad \frac{2 \sin x \cdot \cos 2x}{\sin x \cdot \sin 3x} = \frac{1}{\sin 2x}$$

$$\sin 4x = \sin 3x \quad [\because \sin x \neq 0]$$

$$\therefore 4x = 3x \text{ (not possible, } \because x \neq 0)$$

$$\text{or } 4x = \pi - 3x \Rightarrow 7x = \pi \Rightarrow 7.(\pi/n) = \pi \quad \therefore n = 7.$$